The Eigenvectors of the Right-Justified Pascal Triangle

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Let $R = \left(\binom{i-1}{n-j}\right)_{1\le i,j\le n}$ denote the $n\times n$ matrix formed by right justifying the first n rows of Pascal's triangle. Let a denote the golden ratio $(1+\sqrt{5})/2$. We will show that the eigenvalues of R are $\lambda_j = (-1)^{n+j}a^{2j-n-1}$, $1\le j\le n$, (as conjectured in [1]), with corresponding eigenvectors $\mathbf{u}_j = (u_{ij})_{1\le i\le n}$ where $u_{ij} = \sum_{k=1}^{j}(-1)^{i-k}\binom{i-1}{k-1}\binom{n-i}{j-k}a^{2k-i-1}$. Since the eigenvalues are distinct, the eigenvectors are linearly independent and so form an invertible matrix that diagonalizes R. Scaling the eigenvectors to $\mathbf{v}_j = (-1)^j a^{n-j}/(1+a^2)^{(n-1)/2}\mathbf{u}_j$ yields a diagonalizing matrix V ($V^{-1}RV = \mathrm{diag}(\lambda_j)_{j=1}^n$) with a remarkable property: $V^{-1} = V$. This makes it easy to write down explicit three-summation formulas for the entries of powers of R.

The proofs below that $R\mathbf{u}_p = \lambda_p\mathbf{u}_p$ for $1\le p\le n$ and $V^2 = I_n$ are bracing exercises in manipulating binomial coefficient sums. During these manipulations, summations will be extended over all integers when convenient; recall that a binomial coefficient with a negative lower parameter is zero. We must also be careful to avoid the symmetry trap [2, p. 156]: the symmetry law $\binom{n}{k} = \binom{n}{n-k}$ is valid only when the upper parameter n is nonnegative.

The proof that $R\mathbf{u}_p = \lambda_p\mathbf{u}_p$ uses the (minimal polynomial) equation for the golden ratio: $a^2 = a+1$, equivalently, $1-a = -a^{-1}$, and relies on the following two binomial coefficient identities.

$$\binom{N-J}{K} = \sum_r (-1)^r \binom{N-r}{K-r}\binom{J}{r} \qquad \text{integer } K \quad (*)$$

$$\binom{I}{J}\binom{J}{K} = \binom{I}{K}\binom{I-K}{J-K} \qquad \text{integers } J, K \quad (**)$$

The first follows from the Vandermonde convolution (below) using "upper negation" to write $\binom{N-r}{K-r}$ as $(-1)^{K-r}\binom{K-N-1}{K-r}$, and the second is the "trinomial revision" identity [2, p. 174]. Here goes. The ith entry of $R\mathbf{u}_p$ is

$$\begin{aligned}
&\sum_{j=1}^{n} R_{ij}u_{jp}\\
&= \sum_{j=1}^{n}\binom{i-1}{n-j}\sum_{k=1}^{p}(-1)^{j-k}\binom{j-1}{k-1}\binom{n-j}{p-k}a^{2k-j-1}\\
&\stackrel{1}{=} \sum_{j=1}^{n}\sum_{k=1}^{p}(-1)^{n+j+p+k}\binom{i-1}{j-1}\binom{n-j}{p-k}\binom{j-1}{k-1}a^{2(p+1-k)-(n+1-j)-1}\\
&\stackrel{2}{=} \sum_{j}\sum_{k}(-1)^{n+j+p+k}\binom{i-1}{k-1}\binom{i-k}{j-k}\sum_{r}(-1)^{r}\binom{n-k-r}{p-k-r}\binom{j-k}{r}a^{2p-2k-n+j}\\
&\stackrel{3}{=} \sum_{k}\sum_{r}(-1)^{n+p}\binom{i-1}{k-1}\binom{i-k}{r}\binom{n-k-r}{p-k-r}\left(\sum_{j}\binom{i-k-r}{j-k-r}(-a)^{j-k-r}\right)a^{r-k+2p-n}\\
&\stackrel{4}{=} \sum_{k}\sum_{r}(-1)^{n+p}\binom{i-1}{k-1}\binom{i-k}{r}\binom{n-k-r}{p-k-r}(-a)^{k+r-i}a^{r-k+2p-n}\\
&\stackrel{5}{=} \sum_{r}(-1)^{n+p+r+i+1}\binom{i-1}{r}\left(\sum_{k}(-1)^{k-1}\binom{i-r-1}{k-1}\binom{n-k-r}{p-k-r}\right)a^{2r-i+2p-n}\\
&\stackrel{6}{=} \sum_{r=0}^{i-1}(-1)^{n+p+r+i+1}\binom{i-1}{r}\binom{n-i}{p-1-r}a^{2r-i+2p-n}\\
&\stackrel{7}{=} \sum_{k=1}^{i}(-1)^{n+p+k+i}\binom{i-1}{k-1}\binom{n-i}{p-k}a^{2k-2-i+2p-n}
\end{aligned}$$

and this last sum agrees with $\lambda_p u_{ip}$, as required.

Notes:

1. reverse sum over j and reverse sum over k
2. apply $(*)$ with $N = n-k$ and $J = j-k$ to $\binom{n-j}{p-k}$, and apply $(**)$ to $\binom{i-1}{j-1}\binom{j-1}{k-1}$
3. apply $(**)$ to $\binom{i-k}{j-k}\binom{j-k}{r}$ and rearrange sums
4. apply binomial theorem to sum over j, and use $(1-a)^{i-k-r} = (-a)^{k+r-i}$
5. apply $(**)$ to $\binom{i-1}{k-1}\binom{i-k}{r} = \binom{i-1}{i-k}\binom{i-k}{r}$, and collect terms in k
6. apply $(*)$ to evaluate sum over k
7. change summation index from r to k with $r = k-1$

The proof that $V^2 = I_n$ uses the "trinomial revision" identity (**) and the companion identity

$$\binom{I}{J}\binom{J}{K} = \binom{I}{K}\binom{I-K}{I-J} \qquad \text{integers } I, J, K \quad (***)$$

as well as the following three identities

$$\binom{M}{k}\binom{N}{L-k} = \binom{M+N}{L} \qquad \text{integer } L \quad (\dagger)$$

$$\sum_r (-1)^r \binom{N}{r} = \delta_{N,0} \qquad \text{integer } N \geq 0 \quad (\dagger\dagger)$$

$$\sum_u (-1)^u \binom{N}{L-u}\binom{N-L+u}{u} = \delta_{L,0} \qquad \text{integer } L \quad (\dagger\dagger\dagger)$$

Identity (†) is the Vandermonde convolution, (††) follows from $(1-1)^N = \delta_{N,0}$, (†††) can be reduced to (††) for $N \geq 0$ using (**), and holds for all N since both sides are polynomials in N. Note that we don't need $a^2 = a + 1$: $V^2 = I_n$ holds considering V as a matrix with polynomial entries.

The identity $V^2 = I_n$ is equivalent to $W^2 = (1+a^2)^{n-1} I_n$ with $w_{ij} = (-1)^j a^{n-j} \times \sum_r (-1)^{i-r}\binom{i-1}{r-1}\binom{n-i}{j-r} a^{2r-i-1}$. The equalities in the computation on the next page of the (i,k) entry of W^2 are labelled with the identity used in that step or with a number referring to the following notes.

Notes:

1. reverse sum on j
2. apply the Vandermonde convolution to $\binom{n-j}{s-1}$ and to $\binom{j-1}{k-s}$
3. apply $(***)$ successively to rewrite the first three factors, use the binomial theorem to evaluate the parenthesized sum, and rearrange sums
4. eliminate the sum on s
5. rearrange sums

$$
\begin{aligned}
&\sum_{j=1}^{n} w_{ij} w_{jk} \\
=\quad & \sum_{j=1}^{n}\sum_{r,s}(-1)^{r+s+i+k}\binom{i-1}{r-1}\binom{j-1}{s-1}\binom{n-i}{j-r}\binom{n-j}{k-s}a^{2n+2r+2s-2j-i-k-2} \\
\overset{1}{=}\quad & \sum_{j=1}^{n}\sum_{r,s}(-1)^{r+s+i+k}\binom{i-1}{r-1}\binom{n-j}{s-1}\binom{n-i}{n-j-r+1}\binom{j-1}{k-s}a^{2r+2s+2j-i-k-4} \\
\overset{2}{=}\quad & \sum_{j,r,s,t,u}(-1)^{r+s+i+k}\binom{i-1}{r-1}\binom{r-1}{t-1}\binom{n-j-r+1}{s-t}\binom{n-i}{n-j-r+1}\binom{j+r-i-1}{u}\times \\
& \qquad\qquad \binom{i-r}{k-s-u}a^{2r+2s+2j-i-k-4} \\
\overset{(***)}{=}\quad & \sum_{j,r,s,t,u}(-1)^{r+s+i+k}\binom{i-1}{r-1}\binom{r-1}{t-1}\binom{n-i}{s-t}\binom{n-i-s+t}{j+r-i-1}\binom{j+r-i-1}{u}\times \\
& \qquad\qquad \binom{i-r}{k-s-u}a^{2r+2s+2j-i-k-4} \\
\overset{(**)}{=}\quad & \sum_{r,s,t,u}(-1)^{r+s+i+k}\binom{i-1}{r-1}\binom{r-1}{t-1}\binom{i-r}{k-s-u}\binom{n-i}{s-t}\binom{n-i-s+t}{u}\times \\
& \qquad\qquad \left(\sum_{j}\binom{n-i-s+t-u}{j+r-i-1-u}a^{2(j+r-i-u-1)}\right)a^{2s+i-k+2u-2} \\
\overset{3}{=}\quad & \sum_{s,t,u}(-1)^{s+i+k}\binom{i-1}{t-1}\binom{i-t}{k-s-u}\left(\sum_{r}(-1)^{r}\binom{i-t-k+s+u}{r-t}\right)\times \\
& \qquad\qquad \binom{n-i}{s-t}\binom{n-i-s+t}{u}(1+a^2)^{n-i-s+t-u}a^{2s+i-k+2u-2} \\
\overset{(\dagger\dagger)}{=}\quad & \sum_{\substack{s,t,u\,:\,k-s-u\geq 0\\ i-t=k-s-u}}(-1)^{s+i+k+t}\binom{i-1}{t-1}\binom{i-t}{k-s-u}\times \\
& \qquad\qquad \binom{n-i}{s-t}\binom{n-i-s+t}{u}(1+a^2)^{n-i-s+t-u}a^{2s+i-k+2u-2} \\
\overset{4}{=}\quad & \sum_{t,u\,:\,t\leq i}(-1)^{u}\binom{i-1}{t-1}\binom{n-i}{k-i-u}\binom{n+u-k}{u}(1+a^2)^{n-k}a^{k+2t-i-2} \\
\overset{5}{=}\quad & (1+a^2)^{n-k}a^{k-i}\sum_{t}\binom{i-1}{t-1}a^{2(t-1)}\sum_{u}(-1)^{u}\binom{n-i}{k-i-u}\binom{n+u-k}{u} \\
\overset{(\dagger\dagger\dagger)}{=}\quad & (1+a^2)^{n-k}a^{k-i}(1+a^2)^{i-1}\delta_{i,k} \\
=\quad & (1+a^2)^{n-1}\delta_{i,k}, \quad \text{as required.}
\end{aligned}
$$

Finally, a mild generalization. Let x be an indeterminate and let $R(x)$ be the $n \times n$ matrix with (i, j) entry $\binom{i-1}{n-j} x^{i+j-n-1}$. Then the eigenvalues and eigenvectors of $R(x)$ are precisely as above but with a a root of $a^2 = ax + 1$ rather than of $a^2 = a + 1$.

References

[1] Rhodes Peele and Pantelimon Stănică, Matrix Powers of Column-Justified Pascal Triangles and Fibonacci Sequences, arXiv:math.CO/0010186.

[2] Graham, Knuth, Patashnik, *Concrete Mathematics* (2nd edition), Addison-Wesley, 1989.